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CLASSES OF STRICTLY SINGULAR OPERATORS AND THEIR PRODUCTS*

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ABSTRACT

V. D. Milman proved in [20] that the product of two strictly singular operators on $L_p[0,1]$ $(1 \leqslant p < \infty)$ or on $C[0,1]$ is compact. In this note we utilize Schreier families $\mathcal{S}_\xi$ in order to define the class of $\mathcal{S}_\xi$-strictly singular operators, and then we refine the technique of Milman to show that certain products of operators from this class are compact, under the assumption that the underlying Banach space has finitely many equivalence classes of Schreier-spreading sequences. Finally we define the class of $\mathcal{S}_\xi$-hereditarily indecomposable Banach spaces and we examine the operators on them.

1. Introduction

In this paper we extend the work of V. D. Milman [20] which showed that the product of two strictly singular (bounded linear) operators on $L_p[0,1]$ $(1 \leqslant p < \infty)$ or on $C[0,1]$ is compact. The importance of this fundamental result of V. D. Milman lies in the fact that compact operators are well-understood, unlike strictly singular ones.

First, we use the Schreier families $\mathcal{S}_\xi$ for $1 \leqslant \xi < \omega_1$ which were introduced by D. Alspach and S. A. Argyros [1] and define the classes of $\mathcal{S}_\xi$-strictly singular operators as follows (Definition 2.1 in the main text).

Definition A: If X_1, X_2 are Banach spaces, $T \in \mathcal{L}(X_1, X_2)$ and $1 \leqslant \xi < \omega_1$, we say that T is $\mathcal{S}_\xi$-strictly singular and write $T \in \mathcal{SS}_\xi(X_1, X_2)$ if for every $\varepsilon > 0$ and every basic sequence (x_n) there exist a set $F \in \mathcal{S}_\xi$ and a vector $z \in [x_i]_{i\in F} \setminus \{0\}$, ($[x_i]_{i\in F}$ stands for the closed linear span of $\{x_i\}_{i\in F}$), such that $\|Tz\| \leqslant \varepsilon\|z\|$. If $X_1 = X_2$ then we write $T \in \mathcal{SS}_\xi(X_1)$.

These classes are increasing in ξ (i.e., if $\xi < \zeta$ then every $\mathcal{S}_\xi$-strictly singular operator is an $\mathcal{S}_\zeta$-strictly singular operator). Moreover, they exhaust the class of strictly singular operators defined on separable Banach spaces. In particular we have the following (Theorem 6.5 in the main text).

Theorem B: *Let X be a separable Banach space, Y be a Banach space and $S \in \mathcal{L}(X, Y)$. Then S is strictly singular if and only if S is $\mathcal{S}_\xi$-strictly singular for some $\xi < \omega_1$.*

We also define the notion of Schreier spreading sequence which is closely related to the well-studied notion of spreading model. In fact, every seminormalized basic sequence has a Schreier spreading subsequence. For $1 \leqslant \xi < \omega_1$ we define an equivalence relation $\approx_\xi$ on the set of weakly null Schreier spreading sequences of a Banach space (Definition 3.4). One of the main results of the present paper, which is a refinement of the above mentioned result of V. D. Milman, is our Theorem 4.1. Its statement is slightly stronger than the following simplified version.

Theorem C: *For a Banach space X and an ordinal $1 \leqslant \xi < \omega_1$, if the number of the equivalence classes of the weakly null spreading sequences in X with respect to the equivalence relation $\approx_\xi$ is equal to $n < \infty$, then the product of any $n+1$ $\mathcal{S}_\xi$-strictly singular operators on X is compact.*

We also provide a similar result for products of strictly singular operators. Applications of this result are given to Tsirelson type spaces, Read's space [23] and the invariant subspace problem.

Finally, for $1 \leqslant \xi < \omega_1$, we define the notion of $\mathcal{S}_\xi$-hereditarily indecomposable Banach space as a refinement of the notion of hereditarily indecomposable (HI) Banach space, introduced by W. T. Gowers and B. Maurey [13]. If $\xi < \zeta$, then every $\mathcal{S}_\xi$-HI Banach space is an $\mathcal{S}_\zeta$-HI space, and if X is a separable HI space, then it is $\mathcal{S}_\xi$-HI for some $1 \leqslant \xi < \omega_1$ (Theorem 6.5). The study of operators on complex $\mathcal{S}_\xi$-HI Banach spaces and their subspaces reveals that the $\mathcal{S}_\xi$-strictly singular operators play an analogous role to that strictly singular operators play on the analysis of operators on complex HI spaces. This indicates a potential use of HI spaces towards the solution of the invariant subspace problem (Corollary 6.12).

Schreier families. We recall the definition of the Schreier families $\mathcal{S}_\xi$ (for $1 \leqslant \xi < \omega_1$) which was introduced by D. Alspach and S. A. Argyros [1]. Before defining $\mathcal{S}_\xi$ we recall some general terminology. Let $\mathcal{F}$ be a set of finite subsets of $\mathbb{N}$. We say that $\mathcal{F}$ is **hereditary** if $G \in \mathcal{F}$, whenever $G \subseteq F \in \mathcal{F}$. $\mathcal{F}$ is **spreading** if whenever $\{n_1, n_2, \ldots, n_k\} \in \mathcal{F}$ with $n_1 < n_2 < \cdots < n_k$ and $m_1 < m_2 < \cdots < m_k$ satisfies $n_i \leqslant m_i$ for $i \leqslant k$, then $\{m_1, m_2, \ldots, m_k\} \in \mathcal{F}$. $\mathcal{F}$ is **pointwise closed** if $\mathcal{F}$ is closed in the topology of pointwise convergence in $2^{\mathbb{N}}$. $\mathcal{F}$ is called **regular** if it is hereditary, spreading and pointwise closed. If A and B are two finite subsets of $\mathbb{N}$, then by $A < B$ we mean that $\max A < \min B$.

Similarly, for $n \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, $n \leqslant A$ means $n \leqslant \min A$. We assume that $\varnothing < F$ and $F < \varnothing$ for any non-empty finite set $F \subseteq N$. If $\mathcal{F}$ and $\mathcal{G}$ are regular, then let

$$\mathcal{F}[\mathcal{G}] = \Big\{ \bigcup_1^n G_i \colon n \in \mathbb{N},\ G_1 < \cdots < G_n,\ G_i \in \mathcal{G} \text{ for } i \leqslant n,\ (\min G_i)_1^n \in \mathcal{F} \Big\}.$$

If $\mathcal{F}$ is regular and $n \in \mathbb{N}$, then we define $[\mathcal{F}]^n$ by $[\mathcal{F}]^1 = \mathcal{F}$ and $[\mathcal{F}]^{n+1} = \mathcal{F}\big[[\mathcal{F}]^n\big]$. If F is a finite set, then $\#F$ denotes the cardinality of F. If N is an infinite subset of $\mathbb{N}$, then $[N]^{<\infty}$ denotes the set of all finite subsets of N. For any ordinal number $1 \leqslant \xi < \omega_1$, Schreier families $\mathcal{S}_\xi$ ($\subseteq [\mathbb{N}]^{<\infty}$) are defined as follows: set

$$\mathcal{S}_0 = \big\{\{n\} \colon n \in \mathbb{N}\big\} \cup \{\varnothing\}, \quad \mathcal{S}_1 = \big\{F \subseteq \mathbb{N} \colon \#F \leqslant \min F\big\}.$$

After defining $\mathcal{S}_\xi$ for some $\xi < \omega_1$, set

$$\mathcal{S}_{\xi+1} = \mathcal{S}_1[\mathcal{S}_\xi].$$

If $\xi < \omega_1$ is a limit ordinal and $\mathcal{S}_\alpha$ has been defined for all $\alpha < \xi$, then fix a sequence $\xi_n \nearrow \xi$ and define

$$\mathcal{S}_\xi = \big\{F \colon n \leqslant F \text{ and } F \in S_{\xi_n} \text{ for some } n \in \mathbb{N}\big\}.$$

If $N = \{n_1, n_2, \ldots\}$ is a subsequence of $\mathbb{N}$ with $n_1 < n_2 < \cdots$ and $\mathcal{F}$ is a set of finite subsets of $\mathbb{N}$, then we define $\mathcal{F}(N) = \big\{(n_i)_{i\in F} \colon F \in \mathcal{F}\big\}$. We summarize the properties of the Schreier families that we will need.

Remark 1.1: (i) Each $\mathcal{S}_\xi$ is a regular family.

(ii) $\mathcal{S}_\xi \subseteq \mathcal{S}_{\xi+1}$ for every ξ. However, $\xi < \zeta$ does not generally imply $\mathcal{S}_\xi \subseteq \mathcal{S}_\zeta$.

(iii) Let $1 \leqslant \xi < \zeta < \omega_1$. Then there exists $n \in \mathbb{N}$ so that if $n \leqslant F \in \mathcal{S}_\xi$, then $F \in \mathcal{S}_\zeta$.

(iv) For $n, m \in \mathbb{N}$ we have that $\mathcal{S}_n[\mathcal{S}_m] = \mathcal{S}_{n+m}$. This fails for infinite ordinals. However, the following is true: For all $1 \leqslant \alpha, \beta < \omega_1$ there exist subsequences M and N of $\mathbb{N}$ such that $\mathcal{S}_\alpha[\mathcal{S}_\beta](N) \subseteq \mathcal{S}_{\beta+\alpha}$ and $\mathcal{S}_{\beta+\alpha}(M) \subseteq \mathcal{S}_\alpha[\mathcal{S}_\beta]$. Also for all $1 \leqslant \xi < \omega_1$ and $n \in \mathbb{N}$ there exist subsequences M and N of $\mathbb{N}$ satisfying $[\mathcal{S}_\xi]^n(N) \subseteq \mathcal{S}_{\xi n}$ and $\mathcal{S}_{\xi n}(M) \subseteq [\mathcal{S}_\xi]^n$.

(v) Let $1 \leqslant \beta < \alpha < \omega_1$, $\varepsilon > 0$ and M be a subsequence of $\mathbb{N}$. Then there exists a finite set $F \subseteq M$ and $(a_j)_{j\in F} \subseteq \mathbb{R}^+$ so that $F \in \mathcal{S}_\alpha(M)$, $\sum_{j\in F} a_j = 1$ and if $G \subseteq F$ with $G \in \mathcal{S}_\beta$, then $\sum_{j\in G} a_j < \varepsilon$.

The proofs can be found in [5].

2. Classes of strictly singular operators

Recall that a bounded operator T from a Banach space X to a Banach space Y is called **strictly singular** if its restriction to any infinite-dimensional subspace is not an isomorphism. That is, for every infinite dimensional subspace Z of X and for every $\varepsilon > 0$ there exists $z \in Z$ such that $\|Tz\| < \varepsilon\|z\|$. We say that T is **finitely strictly singular** if for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every subspace Z of X with $\dim Z \geqslant n$ there exists $z \in Z$ such that $\|Tz\| < \varepsilon\|z\|$. In particular, for $1 \leqslant p < q \leqslant \infty$ the inclusion operator $i_{p,q}$ from ℓ_p to ℓ_q is finitely strictly singular. We will denote by $\mathcal{K}(X,Y)$, $\mathcal{SS}(X,Y)$ and $\mathcal{FSS}(X,Y)$ the collections of all compact, strictly singular and finitely strictly singular operators from X to Y, respectively. If $X = Y$ we will write $\mathcal{K}(X)$, $\mathcal{SS}(X)$ and $\mathcal{FSS}(X)$. It is known that these sets are norm closed operator ideals in $\mathcal{L}(X)$, the space of all bounded linear operators on X, see [20, 26] for more details on these classes of operators. It is well-known that $\mathcal{K}(X) \subseteq \mathcal{FSS}(X) \subseteq \mathcal{SS}(X)$. We provide the proof for completeness. The second inclusion is obvious. To prove the first inclusion, suppose that T is not finitely strictly singular. Then there exists $\varepsilon > 0$ and a sequence (E_n) of subspaces of X such that $\dim E_n = n$ and T satisfies $\|Tx\| \geqslant \varepsilon\|x\|$ for each $x \in E_n$. Let $F_n = T(E_n)$. It follows that $\dim F_n = n$ and, for every n and every $y \in T(S_{E_n})$ we have that $\|y\| \geqslant \varepsilon$ (where S_{E_n} denotes the unit sphere of E_n). Let z_1 be in $T(S_{E_1})$. Suppose we have already constructed $z_1, \dots, z_k$ with $z_i \in T(S_{E_i})$ for $i = 1, \dots, k$. Using [17, Lemma 1.a.6] or [11, Lemma of page 2] we can find z_{k+1} in $T(S_{E_{k+1}})$ such that $\operatorname{dist}\big(z_{k+1}, [z_i]_{i=1}^k\big) > \varepsilon/2$. Iterating this procedure we produce a sequence (z_i) in $T(B_X)$ satisfying $\|z_i - z_j\| > \varepsilon/2$ whenever $i \neq j$. It follows that T is not compact.

In this article, we define and study certain classes of strictly singular operators. We also refine certain results about strictly singular operators to the classes of operators that we introduce.

Definition 2.1: If X_1, X_2 are Banach spaces, $T \in \mathcal{L}(X_1, X_2)$ and $1 \leqslant \xi < \omega_1$, we say that T is $\mathcal{S}_\xi$-strictly singular and write $T \in \mathcal{SS}_\xi(X_1, X_2)$ if for every $\varepsilon > 0$ and every basic sequence (x_n) there exist a set $F \in \mathcal{S}_\xi$ and a vector $z \in [x_i]_{i\in F} \setminus \{0\}$, ($[x_i]_{i\in F}$ stands for the closed linear span of $\{x_i\}_{i\in F}$), such that $\|Tz\| \leqslant \varepsilon\|z\|$. If $X_1 = X_2$ then we write $T \in \mathcal{SS}_\xi(X_1)$.

The main difficulty in checking that an operator is $\mathcal{S}_\xi$-strictly singular, seems to be that one has to verify Definition 2.1 for all basic sequences (x_n). Notice that without loss of generality it is enough to check all *normalized* basic sequences. Also notice that if X_1 and X_2 are Banach spaces then $T \in \mathcal{SS}_\xi(X_1, X_2)$ if and only if for every normalized basic sequence (x_n) and $\varepsilon > 0$ there exist a subsequence (x_{n_k}), $F \in \mathcal{S}_\xi$ and $w \in [x_{n_k}]_{k\in F} \setminus \{0\}$ such that $\|Tw\| \leqslant \varepsilon\|w\|$. This is easy to see, since $F \in \mathcal{S}_\xi$ implies that $\{n_k : k \in F\} \in \mathcal{S}_\xi$. For reflexive Banach spaces with bases, we can narrow down this family of basic sequences even more, as the following remark shows.

Remark 2.2: Let $T \in \mathcal{L}(X_1, X_2)$ and $1 \leqslant \xi < \omega_1$. If X_1 is a reflexive Banach space with a basis (e_n), then $T \in \mathcal{SS}_\xi(X_1, X_2)$ if and only if for any normalized block sequence (y_n) of (e_n) and $\varepsilon > 0$ there exists $G \in \mathcal{S}_\xi$ and $w \in [y_n]_{n\in G} \setminus \{0\}$ such that $\|Tw\| \leqslant \varepsilon\|w\|$.

Remark 2.2 follows from the classical fact below. Two basic sequences (x_n) and (y_n) are called C-equivalent for some $C > 1$, denoted by $(x_n) \overset{C}{\approx} (y_n)$, if for every $(a_n) \in c_{00}$ we have that $\|\sum a_n x_n\| \overset{C}{\approx} \|\sum a_n y_n\|$. (We write $a \overset{C}{\approx} b$ if $\frac{1}{C} a \leqslant b \leqslant Ca$.) Two basic sequences (x_n) and (y_n) are called equivalent, denoted by $(x_n) \approx (y_n)$, if they are C-equivalent for some $C \geqslant 1$. Since X_1 is reflexive then every normalized basic sequence (x_n) in X_1 is weakly null and therefore by [7] it has a subsequence (x_{n_k}) which is equivalent to a block sequence (y_k) of (e_n) and $\|x_{n_k} - y_k\| \to 0$.

Remark 2.3: Let (x_n) be a bounded sequence in a Banach space X. Then there is a subsequence (x_{n_k}) such that one of the following conditions hold.

(i) (x_{n_k}) converges;
(ii) (x_{n_k}) is equivalent to the unit vector basis of ℓ_1;
(iii) The difference sequence (d_k) defined by $d_k = x_{n_{2k+1}} - x_{n_{2k}}$ is a seminormalized weakly null basic subsequence. Moreover, if X has a basis, then (d_k) is equivalent to a block sequence of the basis.

This is a standard result. Indeed, if (x_n) has no subsequences satisfying (i) or (ii) then Rosenthal's ℓ_1 Theorem yields a weakly Cauchy subsequence (x_{n_k}). By passing to a further subsequence we may assume that the sequence $(x_{n_{k+1}} - x_{n_k})$ is weakly null and seminormalized. Now (iii) follows by [7].

In view of this Remark 2.3, the requirement "every basic sequence" in Definition 2.1 is "almost" as general as "every sequence".

Proposition 2.4: *Suppose that X and Y are two Banach spaces and $1 \leqslant \xi, \zeta < \omega_1$. Then*

(i) $\mathcal{FSS}(X,Y) \subseteq \mathcal{SS}_\xi(X,Y) \subseteq \mathcal{SS}(X,Y)$.
(ii) *If* $1 \leqslant \xi < \zeta < \omega_1$, *then* $\mathcal{SS}_\xi(X,Y) \subseteq \mathcal{SS}_\zeta(X,Y)$.
(iii) $\mathcal{SS}_\xi(X)$ *is norm-closed;*
(iv) *If* $S \in \mathcal{SS}_\xi(X)$ *and* $T \in \mathcal{L}(X)$, *then* TS *and* ST *belong to* $\mathcal{SS}_\xi(X)$.
(v) *If* $S \in \mathcal{SS}_\xi(X)$ *and* $T \in \mathcal{SS}_\zeta(X)$, *then* $S+T \in \mathcal{SS}_{\xi+\zeta}(X)$. *In particular, if* $S,T \in \mathcal{SS}_\xi(X)$, *then* $S+T \in \mathcal{SS}_{\xi 2}(X)$.

Proof. (i) It is obvious.

(ii) Indeed, for $1 \leqslant \xi < \zeta < \omega_1$, by Remark 1.1(iii) there exists $N \in \mathbb{N}$ such that if $\mathcal{S}_\xi \cap \big[\{N, N+1, \dots\}\big]^{<\infty} \subseteq \mathcal{S}_\zeta$. Now if $T \in \mathcal{SS}_\xi(X,Y)$, ε is a positive number and (x_n) is a normalized basic sequence in X, then consider the basic sequence (y_n) where $y_i = x_{N+i}$. There exists $F \in \mathcal{S}_\xi$ and $z \in [y_i]_{i \in F} \setminus \{0\}$ such that $\|Tz\| \leqslant \varepsilon \|z\|$. Since $F \in \mathcal{S}_\xi$ and $F \subseteq \{N, N+1, \dots\}$ we have that $F \in \mathcal{S}_\zeta$.

(iii) Let $(T_n)_n \subset \mathcal{SS}_\xi(X)$, $T \in \mathcal{L}(X)$ and $\lim_n T_n = T$. Let (x_n) be a seminormalized basic sequence in X and $\varepsilon > 0$. Let $n_0 \in \mathbb{N}$ such that $\|T_{n_0} - T\| \leqslant \varepsilon/2$. Since $T_{n_0} \in \mathcal{SS}_\xi(X)$, there exist $F \in \mathcal{S}_\xi$ and $z \in [x_i]_{i \in F} \setminus \{0\}$ such that $\|T_{n_0} z\| \leqslant \frac{\varepsilon}{2}\|z\|$. Thus

$$\|Tz\| \leqslant \big\|(T_{n_0} - T)z\big\| + \|T_{n_0} z\| \leqslant \frac{\varepsilon}{2}\|z\| + \frac{\varepsilon}{2}\|z\| = \varepsilon\|z\|.$$

(iv) Let $S \in \mathcal{SS}_\xi(X)$ and $T \in \mathcal{L}(X)$. We show that $TS \in \mathcal{SS}_\xi(X)$. Let (x_n) be a basic sequence in X and $\varepsilon > 0$. If $T = 0$ then it is obvious that $TS \in \mathcal{SS}_\xi(X)$. Suppose that $T \neq 0$, then there exist $F \in \mathcal{S}_\xi$ and $z \in [x_n]_{n \in F} \setminus \{0\}$ such that $\|Sz\| \leqslant \frac{\varepsilon}{\|T\|}\|z\|$. Thus, $\|TSz\| \leqslant \|T\|\|Sz\| \leqslant \varepsilon\|z\|$. The proof that $ST \in \mathcal{SS}_\xi(X)$ is due to A. Popov [22] who improved our original argument which only worked in reflexive spaces.

(v) Let $(x_n)_{n \in \mathbb{N}}$ be a normalized basic sequence and $\varepsilon > 0$. By Remark 1.1(iv) let $N = (n_i)$ be a subsequence of $\mathbb{N}$ such that $\mathcal{S}_\zeta[\mathcal{S}_\xi](N) \subseteq \mathcal{S}_{\xi+\zeta}$. Find $F_1 \in \mathcal{S}_\xi$ and $w_1 \in [x_{n_i}]_{i \in F_1}$ such that $\|w_1\| = 1$ and $\|Sw_1\| < \varepsilon/(8C)$ where C is the basis constant of (x_n). Since $(x_{n_i})_{i > F_1}$ is again a basic sequence, we can find $F_2 \in \mathcal{S}_\xi$ and $w_2 \in [x_{n_i}]_{i \in F_2}$ such that $F_1 < F_2$, $\|w_2\| = 1$, and $\|Sw_2\| < \varepsilon/(16C)$. Proceeding inductively we produce sets $F_1 < F_2 < \cdots$ and vectors $w_k \in [x_{n_i}]_{i\in F_k}$ with $\|w_k\| = 1$ and $\|Sw_k\| < \frac{\varepsilon}{2^{k+2}C}$. Since (w_k) is a basic sequence, we find $G \in \mathcal{S}_\zeta$ and $z \in [w_k]_{k\in G}\setminus\{0\}$ such that $\|Tz\| \leqslant \frac{\varepsilon}{2}\|z\|$. Suppose that $G = \{k_1, \dots, k_m\}$ and $z = \sum_{i=1}^m a_i w_{k_i}$. Then we can write $z = \sum_{i\in F} b_i x_{n_i}$ for some $F \in \mathcal{S}_\xi[\mathcal{S}_\xi]$. By the choice of N we have that $z \in [(x_i)_{i\in H}]$ for some $H \in \mathcal{S}_{\xi+\zeta}$. Also, $|a_i| \leqslant 2C\|z\|$. It follows that $\|Sz\| \leqslant \sum_{i=1}^m |a_i|\|Sw_{k_i}\| \leqslant 2C\frac{\varepsilon}{4C}\|z\| = \frac{\varepsilon}{2}\|z\|$, so that $\|(S+T)z\| \leqslant \varepsilon\|z\|$. ∎

Of course, if $1 \leqslant p < q < \infty$ then any bounded operator from ℓ_q to ℓ_p is compact. Also every bounded operator from ℓ_p to ℓ_q is strictly singular, [17].

Example 2.5: If $1 \leqslant p < q < \infty$, then any bounded operator $T \in \mathcal{L}(\ell_p, \ell_q)$ belongs to $\mathcal{SS}_1(\ell_p, \ell_q)$.

If $1 < p$, then we can apply Remark 2.2. Let (x_n) be a normalized block sequence in ℓ_p and $\varepsilon > 0$. If $\inf_i \|Tx_i\|_q = 0$ then we are done (we denote by $\|\cdot\|_p$ and $\|\cdot\|_q$ the norms of ℓ_p and ℓ_q respectively). Hence, assume that (Tx_n) is seminormalized. Since (x_n) is weakly null, (Tx_n) is weakly null. By standard gliding hump arguments [7] we can pass to a subsequence (Tx_{n_i}) such that for some seminormalized block sequence (y_n) in ℓ_q,

$$\Big\|\sum_i a_i T x_{n_i}\Big\|_q \leqslant 2\Big\|\sum_i a_i y_i\Big\|_q \qquad \text{for every } (a_i) \in c_{00}.$$

Hence for $\varepsilon > 0$ one can choose $N \in \mathbb{N}$ such that

$$\Big\|T\Big(\sum_{i=1}^N x_{n_{N+i}}\Big)\Big\|_q \leqslant \varepsilon\Big\|\sum_{i=1}^N x_{n_{N+i}}\Big\|_p.$$

Suppose that $p = 1$. Let (x_n) be a normalized basic sequence in ℓ_1 and $\varepsilon > 0$. By H. P. Rosenthal's ℓ_1 theorem [25] after passing to a subsequence and relabeling we can assume that (x_n) is K-equivalent to the unit vector basis of ℓ_1 for some $K < \infty$.

By applying Remark 2.3 to (Tx_n) there exists a subsequence (x_{n_k}) of (x_n) such that the sequence (d_k) defined by $d_k = Tx_{n_{2k+1}} - Tx_{n_{2k}}$ is either norm null or satisfies (iii) of Remark 2.3. If (d_k) is norm null, then there exists $m \geqslant 2$ such that $\|d_m\| < 2\varepsilon/K$, so that

$$\|Tx_{n_{2m+1}} - Tx_{n_{2m}}\| < \varepsilon \cdot \frac{2}{K} \leqslant \varepsilon\|x_{n_{2m+1}} - x_{n_{2m}}\|.$$

Since $\{n_{2m}, n_{2m+1}\} \in \mathcal{S}_1$ we have $T \in \mathcal{SS}_1(\ell_1, \ell_q)$.

If (d_k) is C-equivalent to a block sequence of the standard basis of ℓ_q then for $\varepsilon > 0$ one can choose $N \in \mathbb{N}$ such that

$$\Big\|T\Big(\sum_{k=1}^{N}(x_{n_{2(N+k)+1}} - x_{n_{2(N+k)}})\Big)\Big\| \leqslant \varepsilon\Big\|\sum_{k=1}^{N}(x_{n_{2(N+k)+1}} - x_{n_{2(N+k)}})\Big\|.$$

Example 2.6: Suppose $1 < p < q < \infty$ with $p \neq q$. Then it is known (see [20, 21, 26]) that $\mathcal{FSS}(\ell_p, \ell_q) \neq \mathcal{SS}(\ell_p, \ell_q) = \mathcal{L}(\ell_p, \ell_q)$. Therefore, Example 2.5 yields $\mathcal{FSS}(\ell_p, \ell_q) \neq \mathcal{SS}_1(\ell_p, \ell_q)$.

Example 2.7: An example of a space X where $\mathcal{SS}_\xi(X) \neq \mathcal{SS}_\zeta(X)$ for some $1 \leqslant \xi < \zeta < \omega_1$.

Fix $1 \leqslant \xi < \omega_1$ consider the space $T[\mathcal{S}_\xi, \frac{1}{2}]$ which is the completion of c_{00} with the norm that satisfies the implicit equation:

$$\|x\|_\xi = \max\Big\{\|x\|_\infty, \sup \frac{1}{2}\sum_i \|E_i x\|_\xi\Big\},$$

where $\|\cdot\|_\infty$ stands for the ℓ_∞ norm, and the supremum is taken for all sets $E_1 < E_2 < \cdots$ such that $(\min E_i)_i \in \mathcal{S}_\xi$.

Since $\xi\omega$ is a limit ordinal, without loss of generality we can assume that the sequence of ordinals in the definition of $\mathcal{S}_{\xi\omega}$ starts with ξ, then $\mathcal{S}_\xi \subseteq \mathcal{S}_{\xi\omega}$ and, therefore, $T[\mathcal{S}_{\xi\omega}, \frac{1}{2}] \subseteq T[\mathcal{S}_\xi, \frac{1}{2}]$. Consider the inclusion operator

$$i_\xi : T[\mathcal{S}_{\xi\omega}, 1/2] \to T[\mathcal{S}_\xi, 1/2].$$

Then $i_\xi \in \mathcal{SS}_{\xi\omega}\big(T[\mathcal{S}_{\xi\omega}, \frac{1}{2}], T[\mathcal{S}_\xi, \frac{1}{2}]\big)$ but $i_\xi \notin \mathcal{SS}_\xi\big(T[\mathcal{S}_{\xi\omega}, \frac{1}{2}], T[\mathcal{S}_\xi, \frac{1}{2}]\big)$.

Indeed, it is easy to verify that $i_\xi \notin \mathcal{SS}_\xi\big(T[\mathcal{S}_{\xi\omega}, \frac{1}{2}], T[\mathcal{S}_\xi, \frac{1}{2}]\big)$, since for every $F \in \mathcal{S}_\xi$ and scalars $(a_i)_{i\in F}$, we have that

$$\Big\|i_\xi\Big(\sum_{i\in F} a_i e_i\Big)\Big\|_\xi = \max\Big\{\max_{i\in F}|a_i|, \frac{1}{2}\sum_{i\in F}|a_i|\Big\} = \Big\|\sum_{i\in F} a_i e_i\Big\|_{\xi\omega},$$

where (e_i) denotes the standard basis of $T[\mathcal{S}_{\xi\omega}, \frac{1}{2}]$.

Now we verify that $i_\xi \in \mathcal{SS}_{\xi\omega}(T[\mathcal{S}_{\xi\omega}, \frac{1}{2}], T[\mathcal{S}_\xi, \frac{1}{2}])$. First recall that $T[\mathcal{S}_{\xi\omega}, \frac{1}{2}]$ is a reflexive Banach space with a basis [3, Proposition 1.1]. Thus we can apply Remark 2.2. Let (x_n) be a normalized block sequence in $T[\mathcal{S}_{\xi\omega}, \frac{1}{2}]$ and $\varepsilon > 0$. If there exists $n \in \mathbb{N}$ such that $\|i_\xi x_n\|_\xi = \|x_n\|_\xi \leqslant \varepsilon$, then we are done. Else assume that $(i_\xi x_n)_n$ is seminormalized. Let $n_i = \min \operatorname{supp} x_i$ (where $\operatorname{supp} x$

stands for the support of the vector x, with respect to (e_i)). By [19, Proposition 4.10] we have that $(i_\xi x_i) \overset{C}{\approx} (e_{n_i})$ where $C := 96 \sup_i \|x_i\|_\xi / \inf_i \|x_i\|_\xi$. We have the following claim which uses the idea and generalizes [3, Proposition 1.5].

Claim 1: For every $\eta > 0$ there exists $F \in \mathcal{S}_{\xi\omega}$ and a convex combination $x := \sum_{i \in F} a_{n_i} e_{n_i}$ such that $\|x\|_\xi < \eta$.

Once Claim 1 is proved, then by letting $\eta := \frac{\varepsilon}{2C}$ it follows that

$$\Big\| i_\xi \Big(\sum_{i \in F} a_{n_i} x_i \Big) \Big\|_\xi = \Big\| \sum_{i \in F} a_{n_i} x_i \Big\|_\xi \leqslant C \|x\|_\xi \leqslant \frac{\varepsilon}{2}$$
$$= \frac{\varepsilon}{2} \sum_{i \in F} a_{n_i} \|x_i\|_{\xi\omega} \leqslant \varepsilon \Big\| \sum_{i \in F} a_{n_i} x_i \Big\|_{\xi\omega}.$$

Thus it only remains to establish Claim 1. For this purpose we need to identity a norming set N^ξ of $T[\mathcal{S}_\xi, \frac{1}{2}]$. We follow [3, p. 976]: Let

$$N_0^\xi = \{\pm e_n^* : n \in \mathbb{N}\} \cup \{0\}.$$

If N_s^ξ has been defined for some $s \in \mathbb{N} \cup \{0\}$, then we define

$$N_{s+1}^\xi = N_s^\xi \cup \Big\{ \frac{1}{2}(f_1 + \cdots + f_d) : f_i \in N_s^\xi,\ (i = 1, \ldots, d),$$
$$\operatorname{supp} f_1 < \operatorname{supp} f_2 < \cdots < \operatorname{supp} f_d \text{ and } (\min \operatorname{supp} f_i)_{i=1}^d \in \mathcal{S}_\xi \Big\}.$$

Finally, set $N^\xi = \bigcup_{s=0}^\infty N_s^\xi$ and the set N^ξ is a norming set for $T[\mathcal{S}_\xi, \frac{1}{2}]$, i.e. we have $\|x\|_\xi = \sup_{x^* \in N^\xi} x^*(y)$ for all $x \in T[\mathcal{S}_\xi, \frac{1}{2}]$.

Now we prove Claim 1. First choose $\ell \in \mathbb{N}$ such that $1/2^\ell < \eta/2$. We have the following claim which follows immediately from Remark 1.1(v).

Claim 2: There exists a convex combination $x = \sum_{i \in F} a_{n_i} e_{n_i}$ such that $F \in \mathcal{S}_{\xi\ell+1} \cap \mathcal{S}_{\xi\omega}$ and $\sum_{i \in G} a_i < \eta/2$ for all $G \in \mathcal{S}_{\xi\ell}$.

Let x as in Claim 2. In order to estimate $\|x\|_\xi$ from above, let $x^* \in N^\xi$. Let $L := \{k \in \mathbb{N} : |x^*(e_k)| \geqslant 1/2^\ell\}$. Then $L \in \mathcal{S}_{\xi\ell}$. Therefore

$$\big|x^*(x)\big| \leqslant \big|(x^*|_L)(x)\big| + \big|(x^*|_{L^c})(x)\big| \leqslant \sum_{k \in L} a_k + 1/2^\ell < \eta/2 + \eta/2 = \eta.$$

This finishes the proof of Claim 1 and the proof that

$$i_\xi \in \mathcal{SS}_{\xi\omega}(T[\mathcal{S}_{\xi\omega}, 1/2], T[\mathcal{S}_\xi, 1/2]).$$

Remark 2.8: Suppose that X and Y are Banach spaces, $1 \leqslant \xi < \omega_1$ and $T \in \mathcal{SS}_\xi(X,Y)$. Let $\widetilde{T} \in \mathcal{L}(X \oplus Y)$ given by $(x,y) \mapsto (0,Tx)$, that is, $\widetilde{T} = \left(\begin{smallmatrix} 0 & 0 \\ T & 0 \end{smallmatrix}\right)$. Then $\widetilde{T} \in \mathcal{SS}_\xi(X \oplus Y)$. Conversely, if $\widetilde{T} \in \mathcal{SS}_\xi(X \oplus Y)$, then $T \in \mathcal{SS}_\xi(X,Y)$.

The converse is obvious. To see the forward implication, pick a normalized basic sequence (x_n, y_n) in $X \oplus Y$ and $\varepsilon > 0$. Since (x_n) is bounded, there exists a subsequence (x_{n_i}) of (x_i) which satisfies one of the options in Remark 2.3. Set $d_k = x_{n_{2k+1}} - x_{n_{2k}}$.

In case (i), $d_m \to 0$, so we can choose m such that $\|d_m\| < \frac{\varepsilon}{C\|T\|}$, where C is the basis constant of (x_n, y_n). Put $h = (x_{n_{2m+1}}, y_{n_{2m+1}}) - (x_{n_{2m}}, y_{n_{2m}})$, then $\operatorname{supp} h = \{n_{2m}, n_{2m+1}\} \in \mathcal{S}_\xi$ and

$$\|\widetilde{T}h\| = \big\|(0, T(x_{n_{2m+1}} - x_{n_{2m}}))\big\| \leqslant \|T\|\|d_m\| < \varepsilon/C \leqslant \varepsilon\|h\|.$$

In case (ii), since $T \in \mathcal{SS}_\xi(X,Y)$ and (x_n) is a basic sequence, we can find $F \in \mathcal{S}_\xi$ and nonzero scalars $(a_i)_{i\in F}$ such that if $w = \sum_{i\in F} a_i x_{n_i}$, then $\|Tw\| \leqslant \varepsilon\|w\|$. Let $h = \sum_{i\in F} a_i (x_{n_i}, y_{n_i})$, then

$$\|\widetilde{T}h\| = \big\|(0, Tw)\big\| \leqslant \varepsilon\|w\| \leqslant \varepsilon\|h\|,$$

where, without loss of generality, we assume that $\big\|(0,y)\big\| = \|y\|$ for all $y \in Y$.

In case (iii), suppose that (d_k) is a basic seminormalized sequence. Then there exists $G \in \mathcal{S}_\xi$ and nonzero scalars $(a_k)_{k\in G}$ such that $\|Tw\| \leqslant \varepsilon\|w\|$ where $w = \sum_{k\in G} a_k d_k$. Set

$$h = \sum_{k\in G} a_k \big((x_{n_{2k+1}}, y_{n_{2k+1}}) - (x_{n_{2k}}, y_{n_{2k}})\big).$$

Then $\|\widetilde{T}h\| = \big\|(0,Tw)\big\| \leqslant \varepsilon\|w\| \leqslant \varepsilon\|h\|$. It is left to show that $\operatorname{supp} h \in \mathcal{S}_\xi$. For a set $A \subseteq \mathbb{N}$ define $A^{\times 2} = \cup_{i\in A}\{2i, 2i+1\}$. By transfinite induction it is easy to see that if $A \in \mathcal{S}_\xi$, then $A^{\times 2} \in \mathcal{S}_\xi$. Thus $F := G^{\times 2} \in \mathcal{S}_\xi$. Therefore, $\operatorname{supp} h = \{n_k \colon k \in F\} \in \mathcal{S}_\xi$ since $\mathcal{S}_\xi$ is spreading Remark 1.1(i).

3. Schreier-spreading sequences and some equivalence relations

Recall the notion of **spreading model**. It is shown in [9, 10] that for every seminormalized basic sequence (y_i) in a Banach space and for every $\varepsilon_n \searrow 0$ there exists a subsequence (x_i) of (y_i) and a seminormalized basic sequence $(\tilde{x}_i)$ (in another Banach space) such that for all $n \in \mathbb{N}$, $(a_i)_{i=1}^n \in [-1,1]^n$ and

$n \leqslant k_1 < \cdots < k_n$ one has

$$\left| \left\| \sum_{i=1}^{n} a_i x_{k_i} \right\| - \left\| \sum_{i=1}^{n} a_i \tilde{x}_i \right\| \right| < \varepsilon_n. \tag{1}$$

The sequence $(\tilde{x}_i)$ is called the **spreading model of** (x_i) and it is a suppression-1 unconditional basic sequence if (y_i) is weakly null. We refer the reader to [9], [10] and [6, I.3. Proposition 2] for more information about spreading models. Spreading models of weakly null seminormalized basic sequences have been studied in [2], where for a Banach space X, the set of all spreading models of all seminormalized weakly null basic sequences of X is denoted by $\mathrm{SP}_w(X)$. Also $\#\mathrm{SP}_w(X)$ denotes the cardinality of the quotient of $\mathrm{SP}_w(X)$ with respect to the equivalence relation $\approx$. In other words, $\#\mathrm{SP}_w(X)$ is the largest number of pair-wise non-equivalent spreading models of weakly null seminormalized basic sequences in X, ([2]).

We will use the following standard fact whose proof is left to the reader.

LEMMA 3.1: *Suppose that (x_n) is a seminormalized basic sequence with a spreading model $(\tilde{x}_n)$. Then, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$\left\| \sum_{i=1}^{n} a_i \tilde{x}_i \right\| \overset{1+\varepsilon}{\approx} \left\| \sum_{i=1}^{n} a_i x_{k_i} \right\|$$

whenever $n_0 \leqslant n \leqslant k_1 < \cdots < k_n$ and $a_1, \ldots, a_n \in \mathbb{R}$.

Motivated by the definition of spreading model we now define the notion of a **Schreier spreading sequence**.

Definition 3.2: Let X be a Banach space. We say that a seminormalized basic sequence (x_n) in X is **Schreier spreading**, if there exists $1 \leqslant C < \infty$ such that for every $F = \{f_1, f_2, \ldots, f_n\}, G = \{g_1, g_2, \ldots, g_n\} \in \mathcal{S}_1$ and scalars $(a_i)_{i=1}^{n}$ we have

$$\left\| \sum_{i=1}^{n} a_i x_{f_i} \right\| \overset{C}{\approx} \left\| \sum_{i=1}^{n} a_i x_{g_i} \right\|.$$

Let $\mathrm{SP}_{1,w}(X)$ denote the set of seminormalized weakly null basic sequences in X which are Schreier spreading (here the index "1" reminds us of $\mathcal{S}_1$, and the index "w" reminds us of *weakly null*).

It follows immediately from the results of Brunel and Sucheston [9, 10] and Lemma 3.1 that:

Remark 3.3: Every seminormalized basic sequence has a Schreier spreading subsequence.

Now for $1 \leqslant \xi < \omega_1$ we define equivalence relations $\approx_\xi$ on $\mathrm{SP}_{1,w}(X)$ as follows.

Definition 3.4: Let X be a Banach space and $1 \leqslant \xi < \omega_1$. Define an equivalence relation $\approx_\xi$ on $\mathrm{SP}_{1,w}(X)$ as follows: if (x_n) and (y_n) are two Schreier spreading sequences in X, we write $(x_n) \approx_\xi (y_n)$ if there exists $1 \leqslant K < \infty$ such that for every $F \in \mathcal{S}_\xi$ and scalars $(a_i)_{i\in F}$ we have that

$$\Big\|\sum_{i\in F} a_i x_i\Big\| \overset{K}{\approx} \Big\|\sum_{i\in F} a_i y_i\Big\|.$$

PROPOSITION 3.5: *Suppose that (x_n) is a Schreier spreading seminormalized basic sequence in X.*

(i) *If (x_{n_k}) is a subsequence of (x_n), then $(x_{n_k})_k$ is Schreier spreading and $(x_n) \approx_1 (x_{n_k})$.*
(ii) *If $(x_n) \approx_1 (y_n)$ for another basic sequence (y_n), then (y_n) is Schreier spreading.*
(iii) *There exists a normalized Schreier spreading sequence (y_n) in X such that $(x_n) \approx (y_n)$.*
(iv) *If X is a reflexive space with a basis (e_n), then there exists a seminormalized block sequence (y_n) of (e_n) such that (y_n) is Schreier spreading and $(x_n) \approx_1 (y_n)$.*

Proof. (i) and (ii) are trivial.

(iii) By standard perturbation arguments, one can find $c_0 \in \big[\inf_n\|x_n\|, \sup_n\|x_n\|\big]$ and a subsequence (x_{n_k}) of (x_n) such that

$$\Big(\frac{x_{n_k}}{\|x_{n_k}\|}\Big) \approx \Big(\frac{x_{n_k}}{c_0}\Big).$$

Hence, if $y_k = \frac{x_{n_k}}{\|x_{n_k}\|}$, then (y_k) is normalized basic Schreier spreading and $(y_k) \approx (x_{n_k}) \approx_1 (x_k)$.

(iv) Since X is reflexive, (x_n) is weakly null. A standard gliding hump argument yields a subsequence (x_{n_k}) of (x_k) and a block sequence (y_k) of (e_k) such that $(x_{n_k}) \approx (y_k)$ which obviously implies the result since $(x_k) \approx_1 (x_{n_k})$. ∎

Corollary 3.6: *For every Banach space X we have*

$$\#SP_w(X) = \#\big(SP_{1,w}(X)/{\approx_1}\big) \leqslant \#\big(SP_{1,w}(X)/{\approx}\big).$$

Proof. The inequality $\#\big(\mathrm{SP}_{1,w}(X)/\approx_1\big) \leqslant \#\big(\mathrm{SP}_{1,w}(X)/\approx\big)$ is obvious. To show that $\#\mathrm{SP}_w(X) = \#\big(\mathrm{SP}_{1,w}(X)/\approx_1\big)$ we define a bijection Φ from the set of $\approx$-equivalence classes of $\mathrm{SP}_w(X)$ to the set of $\approx_1$-equivalence classes of $\mathrm{SP}_{1,w}(X)$. Suppose that $(\tilde{x}_n) \in \mathrm{SP}_w(X)$ is the spreading model of a weakly null seminormalized basic sequence (x_n). Then by Lemma 3.1 there exists $n_0 \in \mathbb{N}$ such that $(x_n)_{n\geqslant n_0} \in \mathrm{SP}_{1,w}(X)$ and $(x_n)_{n\geqslant n_0} \approx_1 (\tilde{x}_n)_{n\in\mathbb{N}}$. Define $\Phi\colon \big((\tilde{x}_n)/{\approx}\big) \mapsto \big((x_n)_{n\geqslant n_0}/{\approx_1}\big)$. Obviously Φ is well-defined and one-to-one. It follows from Remark 3.3 and Proposition 3.5(i) that Φ is onto. ∎

4. Compact products

V. D. Milman [20] proved that the product of any two strictly singular operators in $L_p[0,1]$ $(1 \leqslant p < \infty)$ or $C[0,1]$ is compact. In this section we extend the techniques used by Milman to spaces with finite $\#\big(\mathrm{SP}_{1,w}(X)/{\approx_\xi}\big)$.

Theorem 4.1: *Let X be a Banach space, $1 \leqslant \xi < \omega_1$ and $n \in \mathbb{N} \cup \{0\}$. If $\#\big(SP_{1,w}(X)/\approx_\xi\big) = n$, $S \in \mathcal{SS}(X)$ and $T_1, \ldots, T_n \in \mathcal{SS}_\xi(X)$, then $T_nT_{n-1}\ldots T_1S$ is compact. Moreover, if ℓ_1 does not isomorphically embed in X then $T_nT_{n-1}\ldots T_1$ is compact.*

Furthermore, if $\#\big(SP_{1,w}(X)/\approx\big) = n$ and $T_1, \ldots, T_{n+1} \in \mathcal{SS}(X)$, then $T_{n+1}T_n\ldots T_1$ is compact. Moreover, if ℓ_1 does not isomorphically embed in X, then $T_nT_{n-1}\ldots T_1$ is compact.

Proof. For simplicity, we present the proof in the case $n = 2$. However, it should be clear to the reader how to extend the proof to $n > 2$ or $n = 1$. The case $n = 0$ will be treated at the end. Thus, for the sake of contradiction, suppose that the conclusion of the theorem fails, i.e., T_2T_1S is not compact or $\ell_1 \not\hookrightarrow X$ and T_2T_1 is not compact.

Claim: There exists a seminormalized weakly Cauchy sequence (u_n) such that $(T_2T_1u_n)$ has no convergent subsequences.

If $\ell_1 \not\hookrightarrow X$ and T_2T_1 is not compact, then one can find a normalized sequence (u_n) in X such that $(T_2T_1u_n)$ has no convergent subsequences. By Rosenthal's Theorem [25] we can assume that (u_n) is weakly Cauchy.

Suppose now that T_2T_1S is not compact. Again, find a normalized sequence (v_n) in X such that $(T_2T_1Sv_n)$ has no convergent subsequences. Put $u_n = Sv_n$. Note that $(T_2T_1u_n)$ has no convergent subsequences, so that (u_n) is seminormalized. Apply Rosenthal's Theorem to (v_n). If (v_n) has a weakly Cauchy subsequence, then, by passing to this subsequence, (u_n) is also weakly Cauchy, and we are done. Suppose not, then, by passing to a subsequence and relabeling, we can assume that (v_n) is equivalent to the unit vector basis of ℓ_1. Now apply Rosenthal's Theorem to (u_n). If (u_n) has a subsequence equivalent to the unit vector basis of ℓ_1, then, after passing to this subsequence and relabeling, we would get that the restriction of S to $[v_n]_{n=1}^{\infty}$ is equivalent to an isomorphism on ℓ_1, which contradicts S being strictly singular. Therefore, (u_n) must have a weakly Cauchy subsequence. This completes the proof of the claim.

Since $(T_2T_1u_n)$ has no convergent subsequences, by passing to a subsequence and relabeling, we can assume that $(T_2T_1u_n)$ is ε-separated for some $\varepsilon > 0$. Thus the sequences (x_n), (y_n) and (z_n) are seminormalized, where $x_n := u_{n+1} - u_n$, $y_n := T_1x_n$ and $z_n := T_2T_1x_n$. Since (u_n) is weakly Cauchy, it follows that (x_n), (y_n) and (z_n) are weakly null. By using Corollary 1 of [7] and Remark 3.3, pass to subsequences and relabel in order to assume that (x_n), (y_n) and (z_n) are basic and Schreier spreading.

Since $T_1(x_n) = y_n$ for all n and $T_1 \in \mathcal{SS}_\xi(X)$ we have that $(x_n) \not\approx_\xi (y_n)$. Similarly, since $T_2(y_n) = z_n$ for all n and $T_2 \in \mathcal{SS}_\xi(X)$ we obtain that $(y_n) \not\approx_\xi (z_n)$. Finally by Proposition 2.4 (iv), we have that $T_2T_1 \in \mathcal{SS}_\xi(X)$, so that $(x_n) \not\approx_\xi (z_n)$. Thus $\#\big(\mathrm{SP}_{1,w}(X)/{\approx_\xi}\big) \geqslant 3$, which is a contradiction.

For the "furthermore" statement, if $\#\big(\mathrm{SP}_{1,w}(X)/\approx\big) = 2$, then we can modify the above proof to merely assume that $T_1, T_2 \in \mathcal{SS}(X)$. Notice that since $T_1(x_n) = y_n$ for all n and $T_1 \in \mathcal{SS}(X)$ we have that $(x_n) \not\approx (y_n)$. Indeed, otherwise T_1 induces the restriction operator from $[(x_n)]$ to $[(y_n)]$ via $\sum_{i=1}^{\infty} a_n x_n \mapsto \sum_{i=1}^{\infty} a_n y_n$. This restriction is one-to-one since (y_n) is a basic sequence, and onto since $(x_n) \approx (y_n)$. Hence, the restriction of T to $[x_n]$ would be an isomorphism, contradiction. Similarly, $(y_n) \not\approx (z_n)$ and $(x_n) \not\approx (z_n)$. Thus $\#\big(\mathrm{SP}_{1,w}(X)/{\approx}\big) \geqslant 3$ which is a contradiction.

The statement as well as the proof of this result for $n = 0$ should be given special attention. The assumptions $\#\big(\mathrm{SP}_{1,w}(X)/\approx_\xi\big) = 0$ or $\#\big(\mathrm{SP}_{1,w}(X)/\approx\big) = 0$, combined with Remark 3.3, simply mean that there is no seminormalized weakly null basic sequence in X. The conclusion of the statement if $n = 0$ simply means that $\mathcal{K}(X) = \mathcal{SS}(X)$. In order to verify the result, if $S \in \mathcal{SS}(X)\backslash\mathcal{K}(X)$, then there exists a normalized sequence (v_n) such that (Sv_n) has no convergent subsequence. By Remark 2.3 there is a subsequence (v_{n_k}) such that both (v_{n_k}) and (Sv_{n_k}) are equivalent to the unit vector basis of ℓ_1. This contradicts the assumption that $S \in \mathcal{SS}(X)$. ∎

5. Applications of Theorem 4.1

In this section we give applications and corollaries of Theorem 4.1.

5.1. The first application was obtained by V. D. Milman [20]. From [15] we have that for $2 < p < \infty$, every weakly null seminormalized sequence in $L_p[0,1]$ has a subsequence which is equivalent to the unit vector basis of ℓ_p or ℓ_2. Thus $\#\big(\mathrm{SP}_{1,w}(L_p[0,1])/\approx\big) = 2$. Moreover, $\ell_1 \not\hookrightarrow L_p[0,1]$. Thus, by Theorem 4.1, the product of any two strictly singular operators on $L_p[0,1]$ $(2 < p < \infty)$ is compact.

5.2. An infinite dimensional subspace Y of a Banach space X is said to be **partially complemented** if there exists an infinite dimensional subspace $Z \subset X$ such that $Y \cap Z = 0$ and $Y + Z$ is closed. In general, the adjoint of a strictly singular operator does not have to be strictly singular. However, V. D. Milman proved in [20] that if X^* is separable and every infinite dimensional subspace of X is partially complemented, then the adjoint of every strictly singular operator defined on X is again strictly singular. Milman then used this fact to show that the product of any two strictly singular operators on $L_p[0,1]$ $(1 < p < 2)$ is compact. This can be immediately generalized to the following dual version of Theorem 4.1.

Corollary 5.1: *Suppose that X is a Banach space such that X^* is separable and every infinite dimensional subspace of X is partially complemented. If $\big(SP_{1,w}(X^*)/\approx\big) = n$, and $T_1, \ldots, T_{n+1} \in \mathcal{SS}(X)$, then $T_{n+1}T_n \ldots T_1$ is compact. Moreover, if ℓ_1 does not isomorphically embed in X^*, then $T_nT_{n-1}\ldots T_1$ is compact.*

5.3. Let X be a reflexive Banach space with a basis (e_i) such that for some $1 \leqslant \xi < \omega_1$ there exists $0 < \delta < 1$ such that

$$\Big\|\sum_{i=1}^{n} x_i\Big\| \geqslant \delta \sum_{i=1}^{n}\|x_i\| \tag{2}$$

for any finite block sequence $(x_i)_{i=1}^n$ with $(\min\operatorname{supp} x_i)_{i=1}^n \in \mathcal{S}_\xi$. Fix $m \in \mathbb{N}$ and by Remark 1.1(iv) let $N = (n_i)$ be a subsequence of $\mathbb{N}$ such that $\mathcal{S}_{\xi m}(N) \subseteq [\mathcal{S}_\xi]^m$. Thus for any block sequence (x_n) in X and any $F \in \mathcal{S}_{\xi m}$ we have

$$\Big\|\sum_{i\in F} x_{n_i}\Big\| \geqslant \delta^m \sum_{i\in F}\|x_{n_i}\|. \tag{3}$$

Hence, if (x_{n_i}) is seminormalized, then (x_{n_i}) is $\approx_{\xi m}$-equivalent to the unit vector basis of ℓ_1. Therefore, the proof of Proposition 3.5(iv) gives that if (x_n) is any Schreier spreading sequence in X, then (x_n) is $\approx_{\xi m}$-equivalent to the unit vector basis of ℓ_1. Since $\ell_1 \not\hookrightarrow X$, by Theorem 4.1 we obtain that $\mathcal{SS}_{\xi m}(X) = \mathcal{K}(X)$. Banach spaces that satisfy (2) are for example Tsirelson type spaces $T[\delta, \mathcal{S}_\xi]$ or more general mixed Tsirelson spaces $T\big[\big(\frac{1}{m_i}, \mathcal{S}_{n_i}\big)_{i\in\mathbb{N}}\big]$, or similar type of hereditarily indecomposable Banach spaces constructed and studied in [3].

5.4. Let R be the Banach space, constructed by C. J. Read in [23]. It is shown in [23] that R has precisely two symmetric bases, (which shall be denoted by $(e_m^Y)_n$ and $(e_n^Z)_n$), up to equivalence.

Proposition 5.2: *If (y_n) is a Schreier spreading sequence in R, (not necessarily symmetric and not necessarily a basis for the whole space), then either $(y_n) \approx_1 (e_n^Y)$ or $(y_n) \approx_1 (e_n^Z)$ or (y_n) is $\approx_1$-equivalent to the unit vector basis of ℓ_1.*

Proof. In [23, page 38, lines 14 and 17] two norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$ are constructed on c_{00} so that the standard basis (e_n) of c_{00} is symmetric with respect to either norm [23, page 38, line -2]. Then Y denotes the completion of $(c_{00}, \|\cdot\|_Y)$ and Z denotes the completion of $(c_{00}, \|\cdot\|_Z)$. It is proved in [23, Lemma 2, page 39] that Y and Z are isomorphic (and we denote them by R). Thus if $(e_n^Y)_n$ and $(e_n^Z)_n$ denotes the standard basis of c_{00} in Y and Z respectively, then $(e_n^Y)_n$ and $(e_n^Z)_n$ are normalized symmetric bases for R. Also, $(e_n^Y)_n$ and $(e_n^Z)_n$ are not equivalent by the estimates of [23, page 39, lines 7 and 9]. From page 40, line 13 to the end of Section 6 (page 47) it is shown in [23] that if (y_n) is a symmetric $\|\cdot\|_Y$-normalized block basic sequence of $(e_n^Y)_n$ in R, then (y_n) is equivalent to $(e_n^Y)_n$, or $(e_n^Z)_n$, or the unit vector basis of ℓ_1. (Then, since R is not isomorphic to ℓ_1, R has exactly two symmetric bases.) A closer examination of these pages will reveal that it is actually shown that if (y_n) is any $\|\cdot\|_Y$-normalized block sequence in R, then one of the following two cases is satisfied:

Case 1: (y_n) has a subsequence $(y_{n_i})_i$ which is equivalent to the unit vector basis of ℓ_1 (see [23, page 41, lines 5–7]). Thus if (y_n) is Schreier spreading, then (y_n) is $\approx_1$-equivalent to the unit vector basis of ℓ_1. Moreover, if (y_n) is symmetric ([23, page 41, line 9]), then (y_n) is equivalent to the unit vector basis of ℓ_1.

Case 2: $\lim_{r\to\infty}\big\|\sum_j \lambda_j y_{j+r}\big\|_Y$ is equivalent to either $\big\|\sum_j \lambda_j e_j^Y\big\|_Y$ or $\big\|\sum_j \lambda_j e_j^Z\big\|_Z$ for every $(\lambda_j) \in c_{00}$.

Indeed, $\big\|\sum_j \lambda_j y_{j+r}\big\|_Y$ is the left hand side of the displayed formula [23, page 47, line 7] by virtue of the notation [23, page 40, line 4]. Thus, by [23, page 47, line 7], $\lim_{r\to\infty}\big\|\sum_j \lambda_j y_{j+r}\big\|_Y$ is denoted by $|||\lambda|||$ in [23, page 47, line 10], or by $p(\lambda,\beta)$ in [23, Section 7]. It is concluded in [23, page 50, line 3] that $|||\lambda|||$ is equivalent to either $\big\|\sum_j \lambda_j e_j^Y\big\|_Y$ or $\big\|\sum_j \lambda_j e_j^Z\big\|_Z$.

Thus, in Case 2, if (y_n) is Schreier spreading, then $(y_n) \approx_1 (e_n^Y)$, or $(y_n) \approx_1 (e_n^Z)$. Moreover, if (y_n) is symmetric [23, page 47, line 9], then (y_n) is equivalent to (e_n^Y) or (e_n^Z). ∎

By combining Proposition 5.2 and Theorem 4.1, we obtain that the product of any three operators in $\mathcal{SS}_1(R)$ is compact.

5.5. Theorem 4.1 may also be used to provide invariant subspaces of operators. A well-known theorem of Lomonosov [18] asserts that if T is an operator on a Banach space such that T commutes with a non-zero compact operator, then T has a (proper non-trivial) invariant subspace. Moreover, if the Banach space is over complex scalars and T is not a multiple of the identity, then there exists a proper non-trivial subspace which is hyperinvariant for T. When the Banach space is over real scalars, one can find a hyperinvariant subspace for T if T does not satisfy an irreducible quadratic equation, see [14, 27].

Proposition 5.3: *Suppose that X is a Banach space and $1 \leqslant \xi < \omega_1$.*

(i) *If $\#\big(SP_{1,w}(X)/{\approx_\xi}\big)$ is finite, then every operator $S \in \mathcal{SS}_\xi(X)\setminus\{0\}$ has a non-trivial hyperinvariant subspace.*
(ii) *If $\#\big(SP_{1,w}(X)/{\approx}\big)$ is finite, then every operator $S \in \mathcal{SS}(X)\setminus\{0\}$ has a non-trivial hyperinvariant subspace.*

Proof. Suppose that either $\#\big(\mathrm{SP}_{1,w}(X)/{\approx_\xi}\big)$ is finite and $S \in \mathcal{SS}_\xi(X) \setminus \{0\}$, or $\#\big(\mathrm{SP}_{1,w}(X)/{\approx}\big)$ is finite and $S \in \mathcal{SS}(X) \setminus \{0\}$. If S has eigenvalues, then every eigenspace is a hyperinvariant subspace and we are done. Suppose S has no eigenvalues. So we can assume that S is quasinilpotent with trivial kernel. It follows that S does not satisfy any real-irreducible quadratic equation. Theorem 4.1 implies that S^m is compact for some m. Also, S^m is non-zero as otherwise zero would be an eigenvalue of S. Since S commutes with S^m, it follows that S has a non-trivial hyperinvariant subspace. ∎

A similar reasoning shows that if, under the hypotheses of Proposition 5.3, T commutes with S then T commutes with the compact operator S^m. Therefore, if $S^m \neq 0$ and either X is a complex Banach space or X is real and T does not satisfy any irreducible quadratic equation, then T has a hyperinvariant subspace.

Note that Read [24] constructed an example of a strictly singular operator with no invariant subspaces. A further application of Proposition 5.3 is Corollary 6.12.

6. Hereditarily indecomposable Banach spaces

In [13] an infinite dimensional Banach space was defined to be **hereditarily indecomposable (HI)** if for every two infinite dimensional subspaces Y and Z of X with $Y \cap Z = \{0\}$ the projection from $Y + Z$ to Y defined by $y + z \mapsto y$ (for $y \in Y$ and $z \in Z$) is not bounded. It is is observed in [13] that this is equivalent to the fact that for every two infinite dimensional subspaces Y and Z of X and for every $\varepsilon > 0$ there exists a unit vector $y \in Y$ such that $\operatorname{dist}\big(y, Z\big) < \varepsilon$. This motivates us to introduce the following definition.

Definition 6.1: Let $1 \leqslant \xi < \omega_1$. We say that a Banach space X is $\mathcal{S}_\xi$-hereditary indecomposable (HI_ξ) if for every $\varepsilon > 0$, infinite-dimensional subspace $Y \subseteq X$ and basic sequence (x_n) in X there exist an index set $F \in \mathcal{S}_\xi$ and a unit vector $y \in Y$ such that the $\operatorname{dist}\big(y, [x_i]_{i \in F}\big) < \varepsilon$.

It is obvious that if $1 \leqslant \xi < \omega_1$ and X is HI_ξ, then X is HI. Similarly to Proposition 2.4(ii), if X is HI_ξ and $\xi < \zeta$ then X is HI_ζ.

Remark 6.2: Let X be a Banach space and $1 \leqslant \xi < \omega_1$.

(i) X is HI_ξ if and only if for every $\varepsilon > 0$, infinite-dimensional subspace $Y \subseteq X$ and normalized basic sequence (x_n) in X there exist a subsequence (x_{n_k}), $F \in \mathcal{S}_\xi$ and unit vector $y \in Y$ such that $\mathrm{dist}\big(y, [x_{n_k}]_{k\in F}\big) < \varepsilon$.

(ii) If X is a reflexive Banach space with a basis (e_n), then X is HI_ξ if and only if for every $\varepsilon > 0$, infinite-dimensional block subspace $Y \subseteq X$ and normalized block sequence (y_n) of (e_n), there exists $G \in \mathcal{S}_\xi$ and unit vector $y \in Y$ such that $\mathrm{dist}\big(y, [y_i]_{i\in G}\big) < \varepsilon$.

The proof of (i) is trivial. For the proof of (ii) notice that if X is a reflexive Banach space with a basis (e_n), Y is an infinite dimensional subspace of X and (x_n) is a basic sequence in X, then $(\frac{x_n}{\|x_n\|})$ is weakly null thus by passing to a subsequence and relabeling we can assume that $(\frac{x_n}{\|x_n\|})$ is "near" a block sequence of (e_n) [7]. Similarly, Y contains an infinite dimensional block subspace. The details are left to the reader.

Example 6.3: The HI space constructed by Gowers and Maurey [13], which will be denoted by GM, is an HI_3 space.

Indeed, we outline the proof from [13] that GM is HI and we indicate that the proof actually shows that GM is HI_3. An important building block of the proof is the notion of rapidly increasing sequence vectors (denoted by RIS vectors). Before defining the RIS vectors, we need to back up and define the ℓ_{1+}^n average with constant $1+\varepsilon$ (for $n \in \mathbb{N}$ and $\varepsilon > 0$). Let $n \in \mathbb{N}$ and $\varepsilon > 0$. We say that a vector $y \in GM$ is an ℓ_{1+}^n **average** with constant $1+\varepsilon$ if $\|y\| = 1$ and y can be written as $y = x_1 + \cdots + x_n$ where $x_1 < \cdots < x_n$, all not equal to zero, and $\|x_i\| \leqslant (1+\varepsilon)n^{-1}$ for every i. It is shown in [13, Lemma 3] that if U is any infinite dimensional block subspace of GM, $\varepsilon > 0$ and $n \in \mathbb{N}$, then there exists $y \in U$ which is an ℓ_{1+}^n average with constant $1+\varepsilon$. In fact, the proof shows that (u_n) is a block basis of U then there exists $F \in \mathcal{S}_1$ and $y \in [u_n]_{n\in F}$ which is an ℓ_{1+}^n average with constant $1+\varepsilon$.

For $N \in \mathbb{N}$ and $\varepsilon > 0$ a vector $z \in GM$ is called an **RIS vector** of length N and constant $1+\varepsilon$ if z can be written as $z = (y_1+\cdots+y_N)/\|y_1+\cdots+y_N\|$ where $y_1 < \cdots < y_N$ and each y_k is an $\ell_{1+}^{n_k}$ average with constant $1+\varepsilon$ and the positive integers $(n_k)_{k=1}^N$ are defined inductively to satisfy $n_1 \geqslant 4(1+\varepsilon)2^{N/\varepsilon'}/\varepsilon'$ where $\varepsilon' = \min(\varepsilon, 1)$, and $\sqrt{\log_2(n_{k+1}+1)} \geqslant 2\#\,\mathrm{supp}\, y_k/\varepsilon'$, where $\mathrm{supp}\, y$ stands for the support of the vector y relative to the standard basis of GM. Thus if $N \in \mathbb{N}$, $\varepsilon > 0$ and U is an infinite dimensional block subspace of GM spanned by a block sequence (u_n), then there exists $G \in S_2$ and $z \in [u_n]_{N \in G}$ which is an RIS vector of length N and constant $1+\varepsilon$.

Then the idea of the proof that GM is HI is then the following ([13, page 868]): Given any $k \in \mathbb{N}$ and two block subspaces Y and Z of GM, spanned by block sequences (y_n) and (z_n) respectively, let $x_1 \in Y$ be an RIS of length $M_1 := j_{2k}$ and constant $41/40$ (the sequence (j_n) is an increasing sequence of integers which is used at the definition of the space GM [13, pages 862 and 863]). The vector x_1 determines then a positive integer M_2. Then a vector x_2 is chosen in Z such that $x_1 < x_2$ and x_2 is an RIS vector of length M_2 and constant $41/40$. The vectors x_1 and x_2 determine a positive integer M_3. Then a vector x_2 is chosen in Y such that $x_2 < x_3$ and x_3 is an RIS vector of length M_3 and constant $41/40$. Continue similarly choosing total of k block vectors x_i alternatingly from Y and Z. Let $y = \sum x_{2i-1}/\|\sum x_{2i-1}\| \in Y$ and $z = \sum x_{2i}/\|\sum x_{2i}\| \in Z$. Then it is shown that $\|y+z\| \geqslant (1/3)\sqrt{\log_2(k+1)}\|y-z\|$. Since k is arbitrary, this shows that GM is HI. By the remarks about the support of an RIS vector, one can make sure that there exist $H_1, H_2 \in \mathcal{S}_3$ such that $y \in [y_n]_{n \in H_1}$ and $z \in [z_n]_{n \in H_2}$. This proves that GM is an HI$_3$ space. Proposition 6.11 implies that if GM is considered as a complex Banach space then every operator on GM can be written in the form $\lambda + S$ where $\lambda \in \mathbb{C}$ and $S \in \mathcal{SS}_3(GM)$.

Example 6.4: The HI space constructed by S. A. Argyros and I. Deliyanni [3], which will be denoted by AD, is an HI$_{\omega 3}$ space.

This can be done similarly to the Example 6.3 by closely examining the proof of [3] showing that AD is an HI space.

Next we use Desriptive Set Theory in order to prove the following result which signifies the importance of separable $\mathcal{S}_\xi$-HI Banach spaces and $\mathcal{S}_\xi$-strictly singular operators defined on separable Banach spaces.

Theorem 6.5: *Let X be a separable Banach space, Y be a Banach space and $S \in \mathcal{L}(X, Y)$. Then the following hold.*

(a) *X is HI if and only if X is HI$_\xi$ for some $\xi < \omega_1$.*
(b) *S is strictly singular if and only if S is $\mathcal{S}_\xi$-strictly singular for some $\xi < \omega_1$.*

For the proof of Theorem 6.5 we need some results from Descriptive Set Theory which we briefly recall.

Trees: Let $\mathbb{N}^{<\mathbb{N}}$ by the set of all finite sequences of natural numbers. By $[\mathbb{N}]^{<\mathbb{N}}$ we shall denote the subset of $\mathbb{N}^{<\mathbb{N}}$ consisting of all strictly increasing finite sequences. We view $\mathbb{N}^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. A **tree** T on $\mathbb{N}$ is a downwards closed subset of $\mathbb{N}^{<\mathbb{N}}$. By Tr we denote the set of all trees on $\mathbb{N}$. Thus

$$T \in \mathrm{Tr} \Leftrightarrow \forall s, t \in \mathbb{N}^{<\mathbb{N}} \ (s \sqsubset t \text{ and } t \in T \Rightarrow s \in T).$$

Notice that $[\mathbb{N}]^{<\mathbb{N}}$ belongs to Tr.

By identifying every $T \in \mathrm{Tr}$ with its characteristic function (i.e. an element of $2^{\mathbb{N}^{<\mathbb{N}}}$), it is easy to see that the set Tr becomes a closed subset of $2^{\mathbb{N}^{<\mathbb{N}}}$. For every $\sigma \in \mathbb{N}^{\mathbb{N}}$ and every $n \in \mathbb{N}$ we let $\sigma|n = \big(\sigma(1), \ldots, \sigma(n)\big) \in \mathbb{N}^{<\mathbb{N}}$. A tree $T \in \mathrm{Tr}$ is said to be **well-founded** if for every $\sigma \in \mathbb{N}^{\mathbb{N}}$ there exists $n \in \mathbb{N}$ such that $\sigma|n \notin T$. By WF we denote the subset of Tr consisting of all well-founded trees.

For every $T \in \mathrm{Tr}$, let $T' = \big\{s \in T \colon \exists t \in T \text{ with } s \sqsubset t\big\}$. Observe that $T' \in \mathrm{Tr}$. By transfinite recursion, for every $T \in \mathrm{Tr}$ we define $(T^{(\xi)})_{\xi<\omega_1}$ as follows. We set $T^{(0)} = T$, $T^{(\xi+1)} = \big(T^{(\xi)}\big)'$ and $T^{(\lambda)} = \bigcap_{\xi<\lambda} T^{(\xi)}$ if λ is limit. Notice that $T \in \mathrm{WF}$ if and only if the sequence $(T^{(\xi)})_{\xi<\omega_1}$ is eventually empty. For every $T \in \mathrm{WF}$, the **order** $o(T)$ of T is the least countable ordinal ξ such that $T^{(\xi)} = \varnothing$. We will need the following Boundedness Principle for WF, [16, Theorem 31.2].

Theorem 6.6: *If A is an analytic subset of* WF, *then* $\sup\{o(T) \colon T \in A\} < \omega_1$.

If $S, T \in \mathrm{Tr}$, then a map $\phi : S \to T$ is said to be **monotone** if for every $s_1, s_2 \in S$ with $s_1 \sqsubset s_2$ we have $\phi(s_1) \sqsubset \phi(s_2)$. Notice that if S, T are well-founded trees and there exists a monotone map $\phi : S \to T$, then $o(S) \leqslant o(T)$. Also observe that for every $\xi < \omega_1$ the Schreier family $\mathcal{S}_\xi$ is a well-founded tree and $o(\mathcal{S}_\xi) \geqslant \xi$.

Standard Borel spaces: Let (X, Σ) be a measurable space, i.e. X is a set and Σ is a σ-algebra on X. The pair (X, Σ) is said to be a **standard Borel space** if there exists a Polish topology τ on X such that the Borel σ-algebra of (X, τ) coincides with Σ. Invoking the classical fact that for every Borel subset B of a Polish space (X, τ) there exists a finer Polish topology τ' on X making B clopen and having the same Borel set as (X, τ) (see [16, Theorem 13.1]), we see that if (X, Σ) is a standard Borel space and $B \in \Sigma$, then B equipped with the relative σ-algebra is a standard Borel space too.

Let X be a Polish space and denote by $F(X)$ the set of all closed subsets of X. We endow $F(X)$ with the σ-algebra Σ generated by the sets

$$\big\{F \in F(X) \colon F \cap U \neq \varnothing\big\},$$

where U ranges over all nonempty open subsets of X. The measurable space $\big(F(X), \Sigma\big)$ is called the Effros-Borel space of X. It is well-known that the Effros Borel space is a standard Borel space, [16, Theorem 12.6].

Now let X be a separable Banach space. Denote by $\mathrm{Subs}(X)$ the set of all infinite-dimensional subspaces of X. The set $\mathrm{Subs}(X)$ is a Borel subset of $F(X)$, and so, a standard Borel space on its own right. To see this, notice that by [16, page 79, line+6] the set $\mathcal{A}$ of all closed linear subspaces (finite or infinite dimensional) of X is Borel in $F(X)$. Moreover, by [16, Exercise 12.20], the set $\mathcal{F}$ of all finite-dimensional subspaces of X is also a Borel subset of $F(X)$. As $\mathrm{Subs}(X) = \mathcal{A} \setminus \mathcal{F}$, we conclude that $\mathrm{Subs}(X)$ is Borel. We will need the following fact, which was isolated explicitly in [4]. Its proof follows by a straightforward application of the Kuratowski–Ryll-Nardzewski selection theorem, [16, Theorem 12.13].

Proposition 6.7: *Let X be a separable Banach space. There exists a sequence $S_l : \mathrm{Subs}(X) \to X$, $l \in \mathbb{N}$, of Borel maps such that for every subspace Y of X the sequence $\big(S_l(Y)\big)$ is in the sphere S_Y of Y and, moreover, it is norm dense in S_Y.*

For more background material on $\mathrm{Subs}(X)$ we refer to [4], [8] and [16]. We are ready to proceed to the proof of Theorem 6.5.

Proof of Theorem 6.5. (a) Clearly we only need to show that if X is HI, then X is HI_ξ for some $\xi < \omega_1$. So, fix a separable HI Banach space X. Let

$$\mathcal{B} = \big\{(x_n) \in X^{\mathbb{N}} \colon (x_n) \text{ is a normalized basic sequence in } X\big\}.$$

We claim that $\mathcal{B}$ is F_σ in $X^{\mathbb{N}}$, the later equipped with the product topology. To see this, for every $k \in \mathbb{N}$ let $\mathcal{B}_k$ be the set of all normalized basic sequences (x_n) with basis constant less or equal to k. It is easy to see that $\mathcal{B}_k$ is closed in $X^{\mathbb{N}}$. As $\mathcal{B}$ is the union over all $k \in \mathbb{N}$ of $\mathcal{B}_k$, this shows that $\mathcal{B}$ is F_σ. Since X is separable, $X^{\mathbb{N}}$ is Polish. Thus $\mathcal{B}$ is a standard Borel space.

Let $S_l : \mathrm{Subs}(X) \to X$, $l \in \mathbb{N}$, be the sequence of Borel maps obtained by Proposition 6.7. For every $m \in \mathbb{N}$, every $(x_n) \in \mathcal{B}$ and every $Y \in \mathrm{Subs}(X)$ we define a tree $T = T(m, (x_n), Y) \in \mathrm{Tr}$ to be the set of all $t = (l_1 < \cdots < l_k) \in [\mathbb{N}]^{<\mathbb{N}}$ such that

$$\Big\| S_l(Y) - \sum_{i=1}^{k} a_i x_{l_i} \Big\| \geqslant \frac{1}{m} \quad \text{for any } a_1, \dots, a_k \text{ in } \mathbb{Q} \text{ and any } l \text{ in } \mathbb{N}.$$

For every $m \in \mathbb{N}$ consider the map $\Phi_m : \mathcal{B} \times \mathrm{Subs}(X) \to \mathrm{Tr}$ defined by

$$\Phi_m\big((x_n), Y\big) = T(m, (x_n), Y).$$

Claim 1: The following hold.

(i) For every $m \in \mathbb{N}$ the map Φ_m is Borel.

(ii) For every $m \in \mathbb{N}$, every $(x_n) \in \mathcal{B}$ and every $Y \in \mathrm{Subs}(X)$ the tree $T(m, (x_n), Y)$ is well-founded.

(iii) Let $\zeta < \omega_1$ and assume that X is not HI_ζ. Then there exist $m \in \mathbb{N}$, $(x_n) \in \mathcal{B}$ and $Y \in \mathrm{Subs}(X)$ such that $o(T(m, (x_n), Y)) \geqslant \zeta$.

Proof of the claim. (i) For those readers familiar with descriptive set theoretic computations, this part of the claim is a straightforward consequence of the definition of the tree $T(m, (x_n), Y)$. However, for the convenience of the readers not familiar with these computations, we shall describe a more detailed argument.

Fix $m \in \mathbb{N}$. For every $t \in \mathbb{N}^{<\mathbb{N}}$ let $U_t = \{T \in \mathrm{Tr} \colon t \in T\}$. As the topology on Tr is the pointwise one, we see that the family $\{U_t \colon t \in \mathbb{N}^{<\mathbb{N}}\}$ forms a sub-basis of the topology on Tr. Thus, it is enough to show that for every $t \in \mathbb{N}^{<\mathbb{N}}$ the set

$$\Phi_m^{-1}(U_t) = \Big\{ \big((x_n), Y\big) \in \mathcal{B} \times \mathrm{Subs}(X) \colon t \in T(m, (x_n), Y) \Big\}$$

is Borel. So, let $t \in \mathbb{N}^{<\mathbb{N}}$. If $t \notin [\mathbb{N}]^{<\mathbb{N}}$, then $\Phi_m^{-1}(U_t) = \varnothing$. Hence, we may assume that $t = (l_1 < \cdots < l_k) \in [\mathbb{N}]^{<\mathbb{N}}$.

For every $j \in \mathbb{N}$ the map $\pi_j : \mathcal{B} \times \mathrm{Subs}(X) \to X$ defined by $\pi_j\big((x_n), Y\big) = x_j$ is clearly Borel. For every $\mathbf{a} = (a_i)_{i=1}^k \in \mathbb{Q}^k$ and every $l \in \mathbb{N}$ consider the map $H_{\mathbf{a},l} : \mathcal{B} \times \mathrm{Subs}(X) \to \mathbb{R}$ defined by

$$H_{\mathbf{a},l}\big((x_n), Y\big) = \Big\| S_l(Y) - \sum_{i=1}^{k} a_i x_{l_i} \Big\| = \Big\| S_l(Y) - \sum_{i=1}^{k} a_i \pi_{l_i}\big((x_n), Y\big) \Big\|.$$

Invoking the above remarks, the Borelness of the map S_l and the continuity of the norm, we see that the map $H_{\mathbf{a},l}$ is Borel. Thus, setting $A_{\mathbf{a},l,m} =$

$H^{-1}_{\mathbf{a},l}\big([\frac{1}{m},+\infty)\big)$ we get that $A_{\mathbf{a},l,m}$ is a Borel subset of $\mathcal{B}\times \mathrm{Subs}(X)$ for every $\mathbf{a}\in\mathbb{Q}^k$, every $l\in\mathbb{N}$ and every $m\in\mathbb{N}$. It follows from

$$\Phi_m^{-1}(U_t) = \bigcap_{\mathbf{a}\in\mathbb{Q}^k}\bigcap_{l\in\mathbb{N}} A_{\mathbf{a},l,m}$$

that $\Phi_m^{-1}(U_t)$ is a Borel subset of $\mathcal{B}\times\mathrm{Subs}(X)$, as desired.

(ii) Assume, towards a contradiction, that there exist $m\in\mathbb{N}$, $(x_n)\in\mathcal{B}$ and $Y\in\mathrm{Subs}(X)$ such that the tree $T(m,(x_n),Y)$ is not well-founded. Thus, there exists $\sigma\in\mathbb{N}^{\mathbb{N}}$ such that $\sigma|k\in T(m,(x_n),Y)$ for all $k\in\mathbb{N}$. Set $n_k=\sigma(k)$. Notice that $n_k<n_{k+1}$ for every $k\in\mathbb{N}$. Let $Z=[x_{n_k}]$. As the sequence $\big(S_l(Y)\big)$ is norm dense in S_Y, we see that $\mathrm{dist}(y,Z)\geqslant\frac{1}{m}$ for every $y\in S_Y$. Thus X is not HI, a contradiction.

(iii) Let $\zeta<\omega_1$ such that X is not HI_ζ. By definition, there exist $\varepsilon>0$, $(x_n)\in\mathcal{B}$ and $Y\in\mathrm{Subs}(X)$ such that for every $y\in S_Y$ and every $F\in\mathcal{S}_\zeta$ we have $\mathrm{dist}(y,[x_n]_{n\in F})\geqslant\varepsilon$. Let $m\in\mathbb{N}$ with $\frac{1}{m}<\varepsilon$. It follows that for every $F=\{l_1<\cdots<l_k\}\in\mathcal{S}_\zeta$ we have $F\in T(m,(x_n),Y)$. Hence, the identity map $\mathrm{Id}:\mathcal{S}_\zeta\to T(m,(x_n),Y)$ is a well-defined monotone map, and so $o\Big(T(m,(x_n),Y)\Big)\geqslant o(\mathcal{S}_\zeta)\geqslant\zeta$. The claim is proved.

We set

$$\begin{aligned} A &= \bigcup_{m\in\mathbb{N}}\Phi_m\big(\mathcal{B}\times\mathrm{Subs}(X)\big)\\ &= \Big\{T(m,(x_n),Y)\colon m\in\mathbb{N},(x_n)\in\mathcal{B}\text{ and }Y\in\mathrm{Subs}(X)\Big\}.\end{aligned}$$

By Claim 1(i), we see that A is an analytic subset of Tr. By Claim 1(ii), we get that $A\subseteq\mathrm{WF}$. Hence, by Theorem 6.6, there exists a countable ordinal ξ such that

$$\sup\{o(T):T\in A\}<\xi.$$

Finally, by Claim 1(iii), we conclude that X is HI_ξ, as desired. ∎

(b) The proof is similar to that of part (a). Again it is enough to show that if X is a separable Banach space, Y is a Banach space and $S\in\mathcal{L}(X,Y)$ is strictly singular, then S is $\mathcal{S}_\xi$-strictly singular for some $\xi<\omega_1$. As in the previous part, let $\mathcal{B}$ be the F_σ subset of $X^{\mathbb{N}}$ consisting of all normalized basic sequences in X. For every $m\in\mathbb{N}$ and every $(x_n)\in\mathcal{B}$ we define a tree $T=T\big(m,(x_n)\big)\in\mathrm{Tr}$ to be the set of all $t=(l_1<\cdots<l_k)\in[\mathbb{N}]^{<\mathbb{N}}$ such that

$$\Big\|S\Big(\sum_{i=1}^{k}a_i x_{l_i}\Big)\Big\|\geqslant\frac{1}{m}\Big\|\sum_{i=1}^{k}a_i x_{l_i}\Big\|\quad\text{for any } a_1,\dots,a_k \text{ in } \mathbb{Q}.$$

For every $m\in\mathbb{N}$ consider the map $\Psi_m:\mathcal{B}\to\mathrm{Tr}$ defined by

$$\Psi_m\big((x_n)\big)=T\big(m,(x_n)\big).$$

We have the following analogue of Claim 1. The proof is identical and it is left to the reader.

Claim 2: The following hold.

(i) For every $m\in\mathbb{N}$ the map Ψ_m is Borel.
(ii) For every $m\in\mathbb{N}$ and every $(x_n)\in\mathcal{B}$ the tree $T\big(m,(x_n)\big)$ is well-founded.
(iii) Let $\zeta<\omega_1$ and assume that S is not $\mathcal{S}_\zeta$-strictly singular. Then there exist $m\in\mathbb{N}$ and $(x_n)\in\mathcal{B}$ such that $o\big(T(m,(x_n))\big)\geqslant\zeta$.

By parts (i) and (ii) of Claim 2 and by Theorem 6.6, there exists a countable ordinal ξ such that

$$\sup\{o\big(T(m,(x_n))\big):m\in\mathbb{N}\text{ and }(x_n)\in\mathcal{B}\}<\xi.$$

Hence, by Claim 2(iii), we conclude that S is $\mathcal{S}_\xi$-strictly singular. The proof of the theorem is completed. ∎

Remark 6.8: We notice that part (b) of Theorem 6.5 is not valid if both X and Y are non-separable. To see this, for every $\xi<\omega_1$ let X_ξ and Y_ξ be separable Banach spaces and $T_\xi\in\mathcal{L}(X_\xi,Y_\xi)$ be a strictly singular operator which is not $\mathcal{S}_\xi$-strictly singular and with $\|T_\xi\|=1$. We let

$$X=\Big(\sum_{\xi<\omega_1}\oplus X_\xi\Big)_{\ell_1}\quad\text{and}\quad Y=\Big(\sum_{\xi<\omega_1}\oplus Y_\xi\Big)_{\ell_2}.$$

One can easily "glue" the sequence $(T_\xi)_{\xi<\omega_1}$ to produce a strictly singular operator $T\in\mathcal{L}(X,Y)$ which is not $\mathcal{S}_\xi$-strictly singular for any $\xi<\omega_1$.

In their celebrated paper [13], W. T. Gowers and B. Maurey showed that if X is a complex HI Banach space, then every operator $T\in\mathcal{L}(X)$ can be written as a strictly singular perturbation of a scalar operator. The proof is based on the definition of the infinite singular values of an operator and an important fact that is proved about them. We recall the definition: Let X be a complex

Banach space, $T \in \mathcal{L}(X)$. We say that T is **infinitely singular** if no restriction of T to a subspace of finite codimension is an isomorphism.

LEMMA 6.9 ([13]): *If X is an infinite dimensional Banach space over $\mathbb{C}$ and $T \in \mathcal{L}(X)$ then there exists $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is infinitely singular.*

Using this fact Gowers and Maurey proved the following.

THEOREM 6.10 ([13]): *Every operator on a complex HI space is of the form $\lambda I + S$ where $\lambda \in \mathbb{C}$ and S is strictly singular.*

We use Lemma 6.9 in the proof of the following result which is analogous to Theorem 6.10.

PROPOSITION 6.11: *If $1 \leqslant \xi < \omega_1$ and X is a complex HI$_\xi$ space, then every $T \in \mathcal{L}(X)$ can be written as $T = \lambda I + S$ where $\lambda \in \mathbb{C}$ and $S \in \mathcal{SS}_\xi(X)$.*

Proof. Let X be a complex HI$_\xi$ space and $T \in \mathcal{L}(X)$. Assume that T is not a scalar multiple of the identity, else there is nothing to prove. By Lemma 6.9 there exists $\lambda \in \mathbb{C}$ such that $S = T - \lambda I$ is infinitely singular. We will show that $S \in \mathcal{SS}_\xi(X)$. Let (x_n) be a normalized basic sequence in X and $\varepsilon > 0$. Proposition 2.c.4 of [17] asserts that there is an infinite dimensional subspace Y of X such that $\|S_{|Y}\| < \varepsilon/3$. Since X is HI$_\xi$ there exists $F \in \mathcal{S}_\xi$, a unit vector $y \in Y$ and a vector $x \in [x_n]_{n\in F}$ such that $\|y - x\| < \varepsilon/(3\|S\| + \varepsilon)$. It can then be easily checked that $\|x/(\|x\|) - x\| < \varepsilon/(3\|S\|)$. Hence

$$\begin{aligned}\big\|Sx/\|x\|\big\| &\leqslant \|Sy\| + \|S(y-x)\| + \big\|S\big(x - x/\|x\|\big)\big\| \\ &\leqslant \varepsilon/3 + \|S\|\varepsilon/(3\|S\|) + \|S\|\big\|x - x/\|x\|\big\| \\ &< \varepsilon.\end{aligned}$$

Since $\frac{x}{\|x\|} \in [x_n]_{n\in F}$, we obtain that $S \in \mathcal{SS}_\xi(X)$, which finishes the proof. ∎

Propositions 6.11 and 5.3 yield the following result.

COROLLARY 6.12: *If X is an infinite dimensional complex HI$_\xi$ Banach space for some $1 \leqslant \xi < \omega_1$, such that $\#\big(SP_{1,w}(X)/\approx_\xi\big) < \infty$, then every operator $T \in \mathcal{L}(X)$ which is not a multiple of the identity has a non-trivial hyperinvariant subspace.*

QUESTION: Does there exist any Banach space which satisfies the assumptions of Corollary 6.12?

Finally, we examine operators originating from a subspace of an HI_ξ Banach space X and taking values in X. The next result will be important in their study:

THEOREM 6.13: *If $1 \leqslant \xi < \omega_1$ and X is a HI_ξ Banach space, then $\mathcal{SS}_\xi(X,Y) = \mathcal{SS}(X,Y)$ for every Banach space Y.*

Proof. It follows from Proposition 2.4(i) that $\mathcal{SS}_\xi(X,Y) \subseteq \mathcal{SS}(X,Y)$. Let $T \in \mathcal{SS}(X,Y)$, (x_n) a basic sequence in X, and $0 < \varepsilon < 1$. Choose $\delta > 0$ such that $\frac{\delta(1+\|T\|)}{1-\delta} < \varepsilon$. By Proposition 2.c.4 of [17] there is an infinite dimensional subspace $Z \subseteq X$ such that $\|T_{|Z}\| < \delta$. Since X is HI_ξ, there exists $F \in \mathcal{S}_\xi$ and vectors $x \in [x_n]_{n\in F}$ and $z \in Z$ such that $\|z\| = 1$ and $\|x - z\| < \delta$. It follows that $\|x\| > 1 - \delta$ and

$$\|Tx\| \leqslant \|Tz\| + \|T\|\|x - z\| < \delta\big(1 + \|T\|\big) < \varepsilon\|x\|. \qquad \blacksquare$$

We now extend the following result of V. Ferenczi (which in turn is a generalization of Theorem 6.10).

THEOREM 6.14 ([12]): *If X is a complex HI Banach space, Y is an infinite dimensional subspace of X and $T \in \mathcal{L}(Y,X)$, then there exists $\lambda \in \mathbb{C}$ and $S \in \mathcal{SS}(Y,X)$ such that $T = \lambda i_{Y,X} + S$ where $i_{Y,X} : Y \to X$ is the inclusion map.*

COROLLARY 6.15: *If X is a complex HI_ξ Banach space for some $1 \leqslant \xi < \omega_1$, Y is an infinite dimensional subspace of X and $T \in \mathcal{L}(Y,X)$, then there exists $\lambda \in \mathbb{C}$ and $S \in \mathcal{SS}_\xi(Y,X)$ such that $T = \lambda i_{Y,X} + S$.*

Proof. It follows from Theorem 6.14 that $T = \lambda i_{Y,X} + S$ for some $\lambda \in \mathbb{C}$ and $S \in \mathcal{SS}(Y,X)$. Since a subspace of an HI_ξ-space is again an HI_ξ-space, Theorem 6.13 yields that $S \in \mathcal{SS}_\xi(Y,X)$. $\blacksquare$

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